

TEICHMÜLLER SPACES AS DEGENERATED SYMPLECTIC LEAVES IN DUBROVIN–UGAGLIA POISSON MANIFOLDS

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To Boris Dubrovin in occasion of his sixtieth birthday.

ABSTRACT. In this paper we study the Goldman bracket between geodesic length functions both on a Riemann surface $\Sigma_{g,s,0}$ of genus g with $s = 1, 2$ holes and on a Riemann sphere $\Sigma_{0,1,n}$ with one hole and n orbifold points of order two. We show that the corresponding Teichmüller spaces $\mathcal{T}_{g,s,0}$ and $\mathcal{T}_{0,1,n}$ are realised as real slices of degenerated symplectic leaves in the Dubrovin–Ugaglia Poisson algebra of upper–triangular matrices S with 1 on the diagonal.

1. INTRODUCTION

In this paper we study some special symplectic leaves in the Poisson algebra (1) of upper–triangular matrices S with 1 on the diagonal. This algebra appears as the semi–classical limit of the famous Nelson–Regge algebra in $2 + 1$ -dimensional quantum gravity [27, 28], and in Chern–Simons theory as Fock–Rosly bracket [13]. At classical level, this algebra was discovered in the context of Frobenius manifold theory by Dubrovin and Ugaglia [8, 32] and in the study of non–symmetric bilinear forms by Bondal [1].

In this paper we adopt the isomonodromic deformations perspective. According to Dubrovin’s isomonodromicity theorem part III [8], the metric, the flat coordinates, the pre–potential and the structure constants of a n dimensional semi–simple Frobenius manifold are given by the space of parameters $\mathbf{u} = (u_1, \dots, u_n)$ together with an $n \times n$ skew-symmetric matrix function $V(\mathbf{u})$ such that the linear differential operator

$$\Lambda(z) := \frac{d}{dz} - U - \frac{V(\mathbf{u})}{z}, \quad U = \text{diagonal}(\mathbf{u}),$$

has constant monodromy data as (u_1, \dots, u_n) vary in the configuration space of n points. Generically, the monodromy data of $\Lambda(z)$ are encoded in the so-called *Stokes matrix* S , an upper triangular matrix with 1 on the diagonal.

It turns out that, although the monodromy map

$$V(u) \rightarrow S,$$

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is given by complicated transcendent functions, the Poisson bracket on the space of Stokes matrices is given by very simple quadratic formulae:

$$\begin{aligned}
 (1) \quad & \{s_{ik}, s_{jl}\} = 0, \quad \text{for } i < k < j < l, \\
 & \{s_{ik}, s_{jl}\} = 0, \quad \text{for } i < j < l < k, \\
 & \{s_{ik}, s_{jl}\} = i\pi (s_{ij}s_{kl} - s_{il}s_{kj}), \quad \text{for } i < j < k < l, \\
 & \{s_{ik}, s_{kl}\} = \frac{i\pi}{2} (s_{ik}s_{kl} - 2s_{il}), \quad \text{for } i < k < l, \\
 & \{s_{ik}, s_{jk}\} = -\frac{i\pi}{2} (s_{ik}s_{jk} - 2s_{ij}), \quad \text{for } i < j < k, \\
 & \{s_{ik}, s_{il}\} = -\frac{i\pi}{2} (s_{ik}s_{il} - 2s_{kl}), \quad \text{for } i < k < l.
 \end{aligned}$$

This bracket was obtained in [8] in the case $n = 3$, then for any $n > 3$ in [32], and for this reason it is called the *Dubrovin–Ugaglia* bracket.

The same bracket appeared in Teichmüller theory as the Goldman bracket [16] between geodesic length functions both on a Riemann surface $\Sigma_{g,s,0}$ of genus g with $s = 1, 2$ holes and on a Riemann sphere $\Sigma_{0,1,n}$ with one hole and n orbifold points of order two. Let us denote the two Teichmüller spaces by $\mathcal{T}_{g,s,0}$, $s = 1, 2$ and $\mathcal{T}_{0,1,n}$ respectively. These are *real symplectic manifolds* of dimension respectively

$$\dim_{\mathbb{R}}(\mathcal{T}_{g,s,0}) = \begin{cases} 3n - 7 & \text{for } n \text{ odd} \\ 3n - 8 & \text{for } n \text{ even,} \end{cases} \quad \text{with } g = \left\lfloor \frac{n-1}{2} \right\rfloor, \quad s = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 2 & \text{for } n \text{ even,} \end{cases}$$

and

$$\dim_{\mathbb{R}}(\mathcal{T}_{0,1,n}) = 2(n - 2),$$

while the generic symplectic leaves $\mathcal{L}_{generic}$ in the Dubrovin–Ugaglia bracket have dimension

$$\dim_{\mathbb{C}}(\mathcal{L}_{generic}) = \frac{n(n-1)}{2} - \left\lfloor \frac{n}{2} \right\rfloor.$$

It is natural to ask whether the Teichmüller spaces arise as real slices of some subvarieties of a generic leaf or of a degenerated leaf. In this paper, we prove that in the both cases the Teichmüller spaces correspond to degenerated symplectic leaves whose complex dimension is equal to the real dimension of the Teichmüller space itself (see Theorem 4.1). As a consequence, we give flat coordinates on such degenerated symplectic leaves by introducing a suitable complexification of the shear coordinates.

The paper is organised as follows. In Section 2 we recall some facts about the isomonodromic deformations of the operator $\Lambda(z)$ and about the Dubrovin–Ugaglia bracket. This part is mainly a review, apart perhaps the minor Remark 2.2. In Section 3, we review some basics on Teichmüller theory recalling the characterization of the Stokes matrices whose entries arise as geodesic length functions on a Riemann surface $\Sigma_{g,s,0}$ of genus g with $s = 1, 2$ holes and on a Riemann sphere $\Sigma_{0,1,n}$ with one hole and n orbifold points of order two. Section 4 is original and contains the characterization of the symplectic leaves arising in Teichmüller theory, including the proof of Theorem 4.1 stating that in the both cases (i.e., for the Riemann surface $\Sigma_{g,s,0}$ of genus g with $s = 1, 2$ holes and Riemann sphere $\Sigma_{0,1,n}$ with one hole and n orbifold points of order two) the Teichmüller spaces $\mathcal{T}_{g,s,0}$ and $\mathcal{T}_{0,1,n}$ are the real slices of degenerated symplectic leaves complex dimension equal to the real dimension of the Teichmüller space itself. In subsection 4.1 we discuss an interesting interpretation in terms of a n particle model in Minkowski space and

in subsection 4.4 we discuss the complexification of the shear coordinates. Section 5 contains a heuristic discussion about some minor progress towards the characterisation of the Frobenius manifold structure on the Teichmüller spaces $\mathcal{T}_{g,s,0}$ and $\mathcal{T}_{0,1,n}$.

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2. DUBROVIN–UGAGLIA BRACKET

In this Section, we recall some facts about the monodromy data of the operator $\Lambda(z)$, its monodromy preserving deformations, and the construction of the Dubrovin–Ugaglia bracket by the so-called duality [9] which allows to map $\Lambda(z)$ to a Fuchsian differential operator.

The Dubrovin–Ugaglia bracket is a Poisson bracket on the group of upper-triangular matrices S with 1 on the diagonal. These matrices S arise as *monodromy data* of the following system of n first order ODEs

$$(2) \quad \frac{d}{dz}Y = \left(U + \frac{V}{z} \right) Y$$

where $U = \text{diagonal}(u_1, \dots, u_n)$, $(u_1, \dots, u_n) \in X_n$,

$$X_n = \{(u_1, \dots, u_n) \in \mathbb{C}^n \mid u_i \neq u_j \text{ for } i \neq j\}$$

and $V = -V^T$ is a skew symmetric $n \times n$ matrix with eigenvalues μ_1, \dots, μ_n .

2.1. Monodromy data. A general description of monodromy data of linear systems of ODE can be found in [21, 22, 20]. Here we use the same notations as in [8], where most results of this sub-section are proved.

We fix a real number $\varphi \in [0, 2\pi[$ and consider the open subset $\mathcal{U} \in X_n$ such that the rays L_1, \dots, L_n defined by

$$(3) \quad L_j := \{u_j + i\rho e^{-i\varphi} \mid 0 \leq \rho < \infty\}$$

do not intersect. We assume that the points $(u_1, \dots, u_n) \in \mathcal{U}$ are ordered in such a way that the rays L_1, \dots, L_n exit from infinity in counter-clockwise order.

In [8] it was proved that for a fixed line l

$$l := \{\arg(z) = \varphi\},$$

there exists $\varepsilon > 0$ small enough, $Z \in \mathbb{R}$ large enough, two sectors Π_L and Π_R defined as

$$(4) \quad \begin{aligned} \Pi_R &= \{z : \arg(l) - \pi - \varepsilon < \arg(z) < \arg(l) + \varepsilon, |z| > |Z|\} \\ \Pi_L &= \{z : \arg(l) - \varepsilon < \arg(z) < \arg(l) + \pi + \varepsilon, |z| > |Z|\} \end{aligned}$$

and two unique fundamental solutions $Y_L(z)$ in Π_L and $Y_R(z)$ in Π_R such that

$$(5) \quad Y_{L,R} \sim \left(\mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right) \right) e^{zU}, \quad \text{as } z \rightarrow \infty, \quad z \in \Pi_{L,R}.$$

In the narrow sectors

$$\begin{aligned} \Pi_+ &:= \{z \mid \varphi - \varepsilon < \arg z < \varphi + \varepsilon\} \\ \Pi_- &:= \{z \mid \varphi - \pi - \varepsilon < \arg z < \varphi - \pi + \varepsilon\} \end{aligned}$$

obtained by the intersection of Π_L and Π_R , we have two fundamental matrices with the same asymptotic behaviour (5). They are related by multiplication by a constant invertible matrix

$$Y_L(z) = Y_R(z)S_+, \quad z \in \Pi_+.$$

$$Y_L(z) = Y_R(z)S_-, \quad z \in \Pi_-.$$

The matrices S_+ and S_- are called *Stokes matrices*. Due to the skew symmetry of V , they satisfy the following relation

$$S_-^T = S_+ := S.$$

Thanks to the choice of the order of u_1, \dots, u_n , S is upper triangular with 1 on the diagonal.

Near the regular singular point 0, there exists a fundamental matrix of the system (2) of the form

$$(6) \quad Y_0(z) = (\Gamma + \mathcal{O}(z)) z^\mu z^R, \quad \text{as } z \rightarrow 0,$$

where a branch cut between zero and infinity has been fixed along the negative part l_- of l , the matrix Γ is the eigenvector matrix of V , $V\Gamma = \Gamma\mu$ and R is a nilpotent matrix satisfying the following relation:

$$(7) \quad e^{2\pi i\mu} R = R e^{2\pi i\mu}.$$

The monodromy \mathcal{M}_0 of the system (2) with respect to the normalized fundamental matrix (6) generated by a simple closed loop around the origin is

$$M_0 = \exp(2\pi i\mu) \exp(2\pi iR).$$

The *central connection matrix* C between 0 and ∞ is defined by

$$Y_0(z) = Y_L(z)C, \quad z \in \Pi_L.$$

The *monodromy data* of the system (2) consist of (μ, R, C, S) and are related by

$$(8) \quad C^{-1} S^{-T} S C = \exp(2\pi i\mu) \exp(2\pi iR).$$

2.2. Dual Fuchsian system and its monodromy data. Following [9], we consider a $n \times n$ Fuchsian system of the form

$$(9) \quad \frac{d}{d\lambda} \Phi = \sum_{k=1}^n \frac{A_k}{\lambda - u_k} \Phi,$$

where

$$(10) \quad A_k = E_k \left(\nu - \frac{1}{2} - V \right),$$

and ν is an arbitrary parameter. This system is dubbed *dual* to the system (2). Let us remind how the monodromy data of this system (9) are related to the monodromy data of system (2):

Theorem 2.1. [9] *Let $q = e^{2\pi i\nu}$ and assume that q is not a root of the characteristic equation*

$$(11) \quad \det(qS + S^T) = 0$$

Then there exist n linearly independent solutions $\phi^{(1)}, \dots, \phi^{(n)}$ of the system (9) analytic in $\lambda \in \mathbb{C} \setminus \cup_j L_j$ such that the monodromy transformations M_1, \dots, M_n along the small loops encircling counter-clockwise the points u_1, \dots, u_n are given by

$$(12) \quad M_k = \mathbb{1} - E_k(qS + S^T).$$

The monodromy around infinity is given by $M_\infty = -\frac{1}{q}S^{-1}S^T$.

2.3. Monodromy preserving deformations. The monodromy preserving deformations equations for the system (2) are the following non-linear differential equations

$$(13) \quad \frac{\partial V}{\partial u_i} = [V_i, V], \quad V_i = \text{ad}_{E_i} \text{ad}_U^{-1}(V), \quad i = 1, \dots, n,$$

where E_i is the matrix with entries $E_{i_{kl}} = \delta_{ik}\delta_{il}$. For any solutions $V(u)$ of equation (13), the monodromy data (μ, R, C, S) of the system

$$\frac{d}{dz}Y = \left(U + \frac{V(u)}{z} \right) Y$$

are constant in a disk in X_n . These same equations describe the isomonodromic deformations of (9), namely the monodromy data M_1, \dots, M_n of the system

$$(14) \quad \frac{d}{d\lambda}\Phi = \sum_{k=1}^n \frac{A_k(u)}{\lambda - u_k} \Phi, \quad A_k(u) = E_k\left(\nu - \frac{1}{2} - V(u)\right), \quad k = 1, \dots, n,$$

are constant in a disk in X_n . Indeed equations (13) are equivalent to the Schlesinger equations [30] for A_1, \dots, A_n . In [19] it was proved that the spectral curve of these two systems is the same.

The set of equations (13) can be written as a n -times Hamiltonian system on the space of skew-symmetric matrices V equipped with the standard linear Poisson bracket for $\mathfrak{so}(n) \ni V$:

$$(15) \quad \{V_{ab}, V_{cd}\} = V_{ad}\delta_{bc} + V_{bc}\delta_{ad} - V_{bd}\delta_{ac} + V_{ac}\delta_{bd}.$$

Indeed equation (13) can be rewritten as

$$(16) \quad \frac{\partial V}{\partial u_i} = \{V, H_i\},$$

where the Hamiltonian functions H_i depend on the times u_1, \dots, u_n

$$(17) \quad H_i = \frac{1}{2} \sum_{j \neq i} \frac{V_{ij}^2}{u_i - u_j}.$$

Equivalently the isomonodromic deformations equations for A_1, \dots, A_n can be written as

$$(18) \quad \frac{\partial A_k}{\partial u_i} = \{A_k, H_i\},$$

where $\{\cdot, \cdot\}$ are the standard linear Poisson bracket for $\mathfrak{gl}(n) \ni A_k$ and the Hamiltonian functions H_i are given by:

$$(19) \quad H_i = \frac{1}{2} \sum_{j \neq i} \frac{\text{Tr}(A_i A_j)}{u_i - u_j},$$

and coincide with the previous ones thanks to the fact that

$$\text{Tr}(A_i A_j) = V_{ij}^2.$$

2.4. Korotkin–Samtleben bracket. In this subsection we recall the definition of the Korotkin–Samtleben bracket and obtain the Dubrovin–Ugaglia bracket as its reduction when (11) is satisfied.

According to [24] the standard Lie–Poisson bracket on $\mathfrak{gl}(n, \mathbb{C})$ is mapped by the monodromy map to

$$(20) \quad \begin{aligned} \left\{ M_i \otimes M_i \right\} &= M_i^2 \Omega M_i - M_i^2 \Omega M_i \\ \left\{ M_i \otimes M_j \right\} &= M_i^1 \Omega M_j^2 + M_j^2 \Omega M_i^1 - \Omega M_i^1 M_j^2 - \Omega M_j^2 M_i^1, \quad \text{for } i < j. \end{aligned}$$

This bracket does not satisfy the Jacobi identity, but it reduces to a Poisson bracket on the adjoint invariant objects, i.e., on the traces of the matrices M_1, \dots, M_n and their products.

If q is chosen in such a way that condition (11) is satisfied, the monodromy matrices of the dual Fuchsian system have the form (12) so that

$$(21) \quad \text{Tr}(M_i M_j) = n - 2 - 2q + q S_{ij}^2, \quad i < j,$$

where S_{ij} is the ij entry in the Stokes matrix S . As a consequence the entries of the Stokes matrix S are adjoint invariant, and the Korotkin–Samtleben bracket reduces to a Poisson bracket on them. This was precisely the main idea by Ugaglia, she assumed $q = 1$ and proved that for

$$\det(S + S^T) \neq 0,$$

the restriction of the Korotkin–Samtleben bracket to the entries of the Stokes matrix leads to a closed Poisson algebra given by the formulae (1).

The Casimirs of this Poisson bracket are the eigenvalues of the matrix $S^{-T} S$ so that the generic Poisson leaves $\mathcal{L}_{\text{generic}}$ have dimension

$$\dim(\mathcal{L}_{\text{generic}}) = \frac{n(n-1)}{2} - \left\lfloor \frac{n}{2} \right\rfloor.$$

Remark 2.2. Note that actually it is not necessary to choose $q = 1$. In fact, given any q such that condition (11) is satisfied, it is always true that

$$\begin{aligned} \{s_{ik}, s_{jl}\} &= \frac{1}{q^2 s_{ik} s_{jl}} \{ \text{Tr}(M_i M_k), \text{Tr}(M_j M_l) \} = \\ &= \frac{i\pi (\epsilon(l-k) + \epsilon(k-j) + \epsilon(i-l) + \epsilon(j-i))}{q^2 s_{ik} s_{jl}} \text{Tr}([M_k, M_i][M_j, M_l]), \end{aligned}$$

where $\epsilon(k)$ is the sign of k . By brute force computation, using the specific form of the matrices M_k , one obtains always the same Poisson bracket (1). This observation is quite important when we want to study the case when the rank of the matrix $S + S^T$

is very low. In this case we can pick $q \neq 1$ and prove that the Poisson algebra is (1) anyway. We will discuss this case further in Section 4.

3. POISSON ALGEBRAS OF GEODESIC LENGTH FUNCTIONS

In this Section we discuss the Dubrovin–Ugaglia bracket in the context of Teichmüller theory. We first recall some key facts which will be needed below.

Due to E. Verlinde and H. Verlinde [33] the configuration space of Einstein gravity in $2 + 1$ dimensions is a Riemann surface with boundary components (or holes) and orbifold points times an interval representing the time variable. The algebra of observables is identified with the collection of geodesic length functions of geodesic representatives of homotopy classes of closed curves together with its natural mapping class group action.

The Poisson structure on geodesic length functions is provided by the Goldman brackets [16] and coincides with the Poisson brackets that follow from the Chern–Simons theory [13].

The Poisson algebra of geodesic functions is always closed (and linear) on the subset of geodesic functions corresponding to *multi-curves*, which are sets of curves without intersections and self-intersections. However, these sets are always infinite whereas the Teichmüller spaces $\mathcal{T}_{g,s,n}$ are spaces of (real) dimension $6g - 6 + 2s + 2n$, where g is the genus of the Riemann surface, s is the number of boundary components (or holes) and n is the number of orbifold points.

Multi-curve geodesic functions are therefore algebraically dependent, and one encounters the problem of constructing an algebraically independent (or, at least, finite) basis of observables such that the Poisson brackets become closed on this set. In the general case this problem is still open.

In the special case of Riemann surfaces with one or two holes [5], [6], and in the case of a Riemann sphere with one hole and n orbifold points of order 2 [3], the Poisson algebra generated by the Goldman bracket on geodesic length functions closes and coincides with the Dubrovin–Ugaglia bracket.

Here we recall the basics of this construction, which is based on the graph description of the Teichmüller space. Denote by $\Sigma_{g,s,n}$ a Riemann surface of genus g with s holes and n orbifold points of order two. We assume the hyperbolicity condition $2g - 2 + s > 0$, so that by the Poincaré uniformization theorem, we have

$$\Sigma_{g,s,n} \sim \mathbb{H}_2^+ / \Delta_{g,s,n},$$

where \mathbb{H}_2^+ is the upper half plane and $\Delta_{g,s,n}$ is a Fuchsian group, the fundamental group of the surface $\Sigma_{g,s,n}$:

$$\Delta_{g,s,n} = \langle \gamma_1 \dots, \gamma_{2g+s+n-1} \rangle, \quad \gamma_1 \dots, \gamma_{2g+s+n-1} \in PSL(2, \mathbb{R}).$$

In particular, for orbifold Riemann surfaces, the Fuchsian group $\Delta_{g,s,n}$ is such that all its elements are either hyperbolic or have trace equal to zero.

We recall the Thurston shear-coordinate description [29], [12] of the Teichmüller spaces of Riemann surfaces with holes and, possibly, orbifold points (see [3]). The main idea is to decompose each hyperbolic matrix $\gamma \in \Delta_{g,s,n}$ as a product of the form

$$(22) \quad \gamma = (-1)^K R^{k_{i_p}} X_{Z_{i_p}} \dots R^{k_{i_1}} X_{Z_{i_1}}, \quad i_j \in I, \quad k_{i_j} = 1, 2, \quad K := \sum_{j=1}^p k_{i_j}$$

where I is a set of integer indices and the matrices R , L and X_{Z_i} are defined as follows:

$$R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = -R^2 := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

$$X_{Z_i} := \begin{pmatrix} 0 & -\exp\left(\frac{Z_i}{2}\right) \\ \exp\left(-\frac{Z_i}{2}\right) & 0 \end{pmatrix},$$

and to decompose each traceless element as

$$(23) \quad \gamma_0 = \gamma^{-1} F \gamma,$$

where γ is decomposed as in (22) and

$$F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The main point of this construction is that one can obtain the decompositions (22) and (23) by looking at closed loops on the fat-graph. The fat-graph, or spine, $\Gamma_{g,s,n}$ is a connected graph that can be drawn without self-intersections on $\Sigma_{g,s,n}$ that has all vertices of valence three except exactly n one-valent vertices situated at the orbifold points, has a prescribed cyclic ordering of labeled edges entering each vertex, and it is a maximal graph in the sense that its complement on the Riemann surface is a set of disjoint polygons (faces), each polygon containing exactly one hole (and becoming simply connected after gluing this hole). Since a graph must have at least one face, only Riemann surfaces with holes, $s > 0$, can be described in this way. These fat graphs (or spines) constructed originally in [12] [14] in the case of surfaces without orbifold points are dual to ideal triangle decompositions of Penner [29].

We obtain the decomposition of an element of the Fuchsian group $\Delta_{g,s,n}$ using the one-to-one correspondence between closed paths in the fat graph (spine) $\Gamma_{g,s,n}$ and conjugacy classes of the Fuchsian group $\Delta_{g,s,n}$. The decomposition (22) can be obtained by establishing a one-to-one correspondence between elements of the Fuchsian group itself and closed paths in the spine starting and terminating at the same directed edge. Each time the path A corresponding to the element γ_A (or, equivalently, to its invariant closed geodesic) passes through the α th edge, an edge-matrix X_{Z_α} with the real coordinate Z_α (related to the length of that edge) appears in the decomposition of γ . At the end of the edge, the path can either turn right or left, and a matrix R or L respectively appears in the decomposition [12]. To obtain decomposition (23), we observe that when a path reaches a one-valent vertex (a pending vertex), it undergoes an *inversion* [2], which corresponds to inserting the matrix F into the corresponding string of 2×2 -matrices. The edge terminating at a pending vertex is called a *pending* edge.

The algebras of geodesic length functions were constructed in [2] by postulating the Poisson relations on the level of the shear coordinates Z_α of the Teichmüller space:

$$(24) \quad \{f(\mathbf{Z}), g(\mathbf{Z})\} = \sum_{\substack{\text{3-valent} \\ \text{vertices } \alpha = 1}}^{4g+2s+n-4} \sum_{i=1}^{3 \bmod 3} \left(\frac{\partial f}{\partial Z_{\alpha_i}} \frac{\partial g}{\partial Z_{\alpha_{i+1}}} - \frac{\partial g}{\partial Z_{\alpha_i}} \frac{\partial f}{\partial Z_{\alpha_{i+1}}} \right),$$

where the sum ranges all the *three-valent* vertices of a graph and α_i are the labels of the cyclically (counterclockwise) ordered ($\alpha_{i+3} \equiv \alpha_i$) edges incident to the vertex with the label α . This bracket gives rise to the *Goldman bracket* on the space of geodesic length functions [16].

In terms of geodesic length functions the bracket (24) corresponds to

$$(25) \quad \{\text{Tr} \gamma_A, \text{Tr} \gamma_B\} = \frac{1}{2} \text{Tr}(\gamma_A \gamma_B) - \frac{1}{2} \text{Tr}(\gamma_A \gamma_B^{-1}).$$

So we see that every time we consider the bracket between the geodesics lengths of two loops A and B , we produce the geodesics lengths of two new loops AB and AB^{-1} . To close the Poisson algebra one must use the *skein relation* valid for two arbitrary matrices in $PSL(2)$:

$$(26) \quad \text{Tr} \gamma_A \text{Tr} \gamma_B = \text{Tr}(\gamma_A \gamma_B) + \text{Tr}(\gamma_A \gamma_B^{-1}).$$

We can use this relation for resolving the crossing between the two geodesics A and B as in Fig. 1.

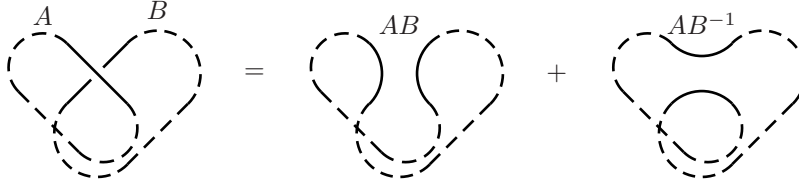


FIGURE 1. The classical skein relation.

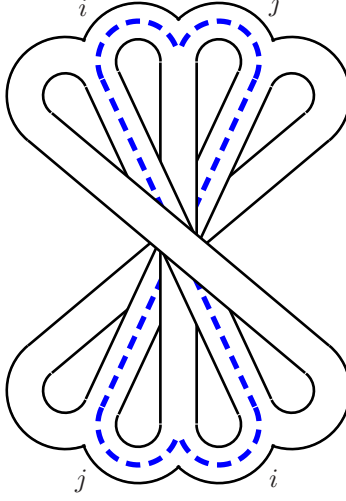
The skein relation is often not enough to close the Poisson algebra on a finite set of generators. In this paper, we shall consider two special cases in which we indeed can close the algebra just by means of skein relation: the case of Riemann surfaces of genus g and one or two holes (which we dub *CFP* due to the fact that it was mostly developed in [6, 4]) in subsection 3.1 and the case of a Riemann sphere with one hole and with $n \geq 3$ orbifold points of order two (dubbed \mathcal{A}_n case due to its close ties to cluster algebra theory [15]) in subsection 3.2.

3.1. The CFP case. This is the case of a Riemann surface of genus g with one or two holes, the fat-graph on which graph-simple geodesics constitute a convenient algebraic basis is shown in Fig. 2. The genus $g = \lfloor \frac{n-1}{2} \rfloor$, where n is the number of vertical edges, and the number s of holes is

$$s = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 2 & \text{for } n \text{ even.} \end{cases}$$

Graph-simple closed geodesics in this picture are those and only those that pass through exactly two different vertical edges; we can then enumerate them by ordered pairs of edge indices denoting by G_{ij} ($i < j$) the corresponding geodesic functions. Denoting by Z_1, \dots, Z_n the coordinates on the vertical edges and by Y_1, \dots, Y_{2n-6} those on the horizontal edges, we obtain

$$(27) \quad G_{ij} = X_{Z_i} L Y_{n+i-4} \dots R X_{Y_{n+j-5}} L X_{Z_j} R X_{Y_{j-2}} \dots X_{Y_i} L X_{Y_{i-1}} R,$$

FIGURE 2. The fat graph in the *CFP* case, the blue geodesic is $G_{i,j}$.

so, for example,

$$\begin{aligned}
 G_{12} &= X_{Z_1} L X_{Z_2} R, \\
 G_{13} &= X_{Z_1} R X_{Y_{n-2}} L X_{Z_3} R X_{Y_1} L, \\
 &\dots \\
 G_{1n} &= X_{Z_1} R X_{Y_{n-2}} R X_{Y_{n-1}} \dots R X_{Y_{2n-6}} L X_{Z_n} L X_{Y_{n-3}} L \dots X_{Y_1} L, \\
 G_{23} &= X_{Z_2} L X_{Y_{n-2}} L X_{Z_3} R X_{Y_1} R.
 \end{aligned}$$

The Poisson algebra for the functions G_{ij} is described by

$$(28) \quad \{G_{ij}, G_{kl}\} = \begin{cases} 0, & j < k, \\ 0, & k < i, \ j < l, \\ G_{ik} G_{jl} - G_{kj} G_{il}, & i < k < j < l, \\ \frac{1}{2} G_{ij} G_{jl} - G_{il}, & j = k, \\ G_{il} - \frac{1}{2} G_{ij} G_{il}, & i = k, \ j < l \\ G_{ik} - \frac{1}{2} G_{ij} G_{kj}, & j = l, \ i < k. \end{cases}$$

This is just a rescaled Dubrovin–Ugaglia bracket .

3.2. The \mathcal{A}_n case. The simplest case of orbifold Riemann surface is a Riemann sphere $\Sigma_{0,1,n}$ with one hole and $n \geq 3$ orbifold points of order two. In this case, the fat-graph $\Gamma_{0,1,n}$ is a tree-like graph with n pending vertices depicted in Fig. 3 for $n = 3, 4$. We enumerate the n pending vertices counterclockwise, $i, j = 1, \dots, n$, and consider the algebra of all geodesic functions.

We consider a basis $\gamma_1, \dots, \gamma_n$ in the Fuchsian group $\Delta_{0,1,n}$ such that

$$(29) \quad -\text{Tr}(\gamma_i \gamma_j) = G_{i,j}.$$

(The sign convention is such that when we interpret $G_{i,j}$ as being the geodesic functions related to lengths $\ell_{i,j}$ of closed geodesics, we have $G_{i,j} = 2 \cosh(\ell_{i,j}/2) \geq 2$.) In this case, for convenience we let Z_i denote the coordinates of pending edges

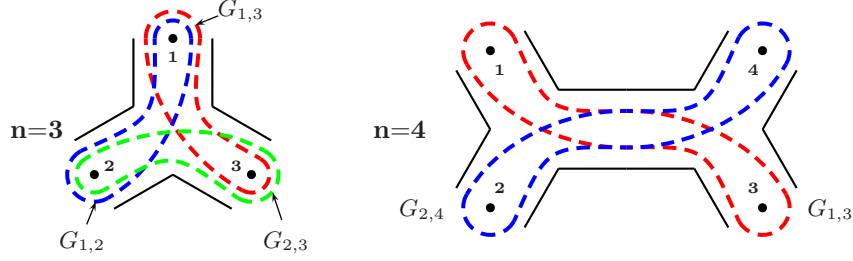


FIGURE 3. Generating graphs for \mathcal{A}_n algebras for $n = 3, 4$. We indicate character geodesics whose geodesic functions G_{ij} enter bases of the corresponding algebras.

and Y_j all other coordinates. This basis in the Fuchsian group $\Delta_{0,1,n}$ is given by the following:

$$\begin{aligned}
 \gamma_1 &= F, \\
 \gamma_2 &= -X_{Z_1} L X_{Z_2} F X_{Z_2} R X_{Z_1} \\
 \gamma_3 &= -X_{Z_1} R X_{Y_1} L X_{Z_3} F X_{Z_3} R X_{Y_1} L X_{Z_1} \\
 &\dots \\
 (30) \quad \gamma_i &= -X_{Z_1} R X_{Y_1} R X_{Y_2} \dots R X_{Y_{i-2}} L X_{Z_i} F X_{Z_i} R X_{Y_{i-2}} L \dots X_{Y_1} L X_{Z_1}, \\
 &\dots \\
 \gamma_{n-1} &= -X_{Z_1} R X_{Y_1} R X_{Y_2} \dots R X_{Y_{n-3}} L X_{Z_{n-1}} F X_{Z_{n-1}} R X_{Y_{n-3}} L \dots X_{Y_1} L X_{Z_1}, \\
 \gamma_n &= -X_{Z_1} R X_{Y_1} R X_{Y_2} \dots R X_{Y_{n-3}} R X_{Z_n} F X_{Z_n} R X_{Y_{n-3}} L \dots X_{Y_1} L X_{Z_1},
 \end{aligned}$$

Observe that $\text{Tr} \gamma_i = 0$, $i = 1, \dots, n$. It is not hard to check that the matrix

$$\gamma_\infty := (\gamma_1 \gamma_2 \dots \gamma_n)^{-1}$$

has eigenvalues $(-1)^{n-1} e^{\pm P/2}$, where P is the length of the perimeter around the hole:

$$(31) \quad P = 2 \sum_{i=1}^n Z_i + 2 \sum_{j=1}^{n-3} Y_j.$$

Let $G_{i,j} = -\text{Tr}(\gamma_i \gamma_j)$ with $i < j$ denote the geodesic function corresponding to the geodesic line that encircles exactly two pending vertices with the indices i and j . Examples for $n = 3$ and $n = 4$ are in figure 3. It turns out that these geodesic functions suffice for closing the Poisson algebra:

$$\begin{aligned}
 \{G_{i,k}, G_{j,l}\} &= 0, \quad \text{for } i < k < j < l, \quad \text{and for } i < j < l < k, \\
 \{G_{i,k}, G_{j,l}\} &= 2(G_{i,j} G_{k,l} - G_{i,l} G_{k,j}), \quad \text{for } i < j < k < l, \\
 (32) \quad \{G_{i,k}, G_{k,l}\} &= G_{i,k} G_{k,l} - 2G_{i,l}, \quad \text{for } i < k < l, \\
 \{G_{i,k}, G_{j,k}\} &= -(G_{i,k} G_{j,k} - 2G_{i,j}), \quad \text{for } i < j < k, \\
 \{G_{i,k}, G_{i,l}\} &= -(G_{i,k} G_{i,l} - 2G_{k,l}), \quad \text{for } i < k < l.
 \end{aligned}$$

Note that this is again a simple rescaling of the Dubrovin–Ugaglia bracket.

Remark 3.1. The formulae for G_{ij} in terms of the shear coordinates $Z_1, \dots, Z_n, Y_1, \dots, Y_{n-3}$ in the \mathcal{A}_n case coincide with a specialization of the formulae of the

geodesics G_{ij} given by (27) in which we assume $Y_{n-3+i} = Y_i$ for $i = 1, \dots, n-3$ and we take the double lengths $2Z_1, \dots, 2Z_n$. In other words:

$$G_{ij}^{(A_n)}(Z_1, \dots, Y_{n-3}) = G_{ij}^{CFP}(2Z_1, \dots, 2Z_n, Y_1, \dots, Y_{n-3}, Y_1, \dots, Y_{n-3}).$$

4. SYMPLECTIC LEAVES CORRESPONDING TO THE TEICHMÜLLER SPACE

As mentioned in the introduction, since the Poisson bracket for geodesic length functions both in the CFP case (Riemann surface $\Sigma_{g,s,0}$ of genus g with $s = 1, 2$ holes) and in the \mathcal{A}_n case (Riemann sphere $\Sigma_{0,1,n}$ with one hole and n orbifold points of order two) coincides with the Dubrovin–Ugaglia bracket, it makes sense to characterize the symplectic leaves to which the two Teichmüller spaces $\mathcal{T}_{g,s,0}$, $s = 1, 2$, and $\mathcal{T}_{0,1,n}$ belong.

In particular we recall that

$$\dim_{\mathbb{R}}(\mathcal{T}_{g,s,0}) = \begin{cases} 3n-7 & \text{for } n \text{ odd} \\ 3n-8 & \text{for } n \text{ even,} \end{cases} \quad \text{where } g = \left\lfloor \frac{n-1}{2} \right\rfloor, \quad s = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 2 & \text{for } n \text{ even,} \end{cases}$$

and

$$\dim_{\mathbb{R}}(\mathcal{T}_{0,1,n}) = 2(n-2),$$

while the generic symplectic leaves $\mathcal{L}_{generic}$ in the Dubrovin–Ugaglia bracket have dimension

$$\dim_{\mathbb{C}}(\mathcal{L}_{generic}) = \frac{n(n-1)}{2} - \left\lfloor \frac{n}{2} \right\rfloor.$$

It is natural to ask whether the Teichmüller spaces arise as real slices of some subvarieties of a generic leaf or of a degenerated leaf. In this section we prove that in both cases the Teichmüller spaces correspond to degenerated symplectic leaves complex dimension equal to the real dimension of the Teichmüller space itself:

Theorem 4.1. *Denote by $\mathcal{L}_{\mathcal{A}_n}$ and by \mathcal{L}_{CFP} the symplectic leaves to which the Stokes matrices of with entries $s_{ij} = G_{ij}$ where G_{ij} are given respectively by (29,30) or by (27) belong. Then*

$$\dim_{\mathbb{C}}(\mathcal{L}_{CFP}) = \begin{cases} 3n-7 & \text{for } n \text{ odd} \\ 3n-8 & \text{for } n \text{ even} \end{cases}$$

and

$$\dim_{\mathbb{C}}(\mathcal{L}_{\mathcal{A}_n}) = 2(n-2),$$

Proof. In order to compute the dimension of the symplectic leaf to which a particular Stokes matrix belongs we use a formula by Bondal [1] which is based on the block diagonal form of the Jordan normal form J_0 of $S^{-T}S$:

Lemma 4.2. *Given an arbitrary upper triangular matrix S with 1 on the diagonal, the Jordan normal form J_0 of $S^{-T}S$ decomposes as follows*

$$(33) \quad J_0 = \sum_{\lambda \neq (-1)^{k+1}} n_{\lambda,k} \left(J_{\lambda,k} \oplus J_{\frac{1}{\lambda},k} \right) + \sum_{\lambda = (-1)^{k+1}} m_{\lambda,k} J_{\lambda,k},$$

where $J_{\lambda,k}$ denotes the $k \times k$ Jordan block with eigenvalue λ , i.e.

$$J_{\lambda,k} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda & 1 \\ 0 & \dots & 0 & 0 & \lambda \end{pmatrix},$$

and $n_{\lambda,k}$ and $m_{(-1)^{k+1},k}$ are the multiplicities of the blocks $J_{\lambda,k} \oplus J_{\frac{1}{\lambda},k}$ and $J_{(-1)^{k+1},k}$ respectively.

The dimension of the symplectic leaf \mathcal{L}_S to which S belongs is

$$\dim_{\mathbb{C}}(\mathcal{L}_S) = \frac{n(n-1)}{2} - d(S),$$

where

$$(34) \quad \begin{aligned} d(S) = & \sum_{\lambda \neq \pm 1} \min(k, l) n_{\lambda,k} n_{\lambda,l} + 2 \sum_{\lambda = \pm 1} \min(k, l) n_{\lambda,k} n_{\lambda,l} + \\ & + 2 \sum_{\lambda = \pm 1} \min(k, l) n_{\lambda,k} m_{\lambda,l} + \frac{1}{2} \sum_{\lambda = \pm 1} \min(k, l) m_{\lambda,k} m_{\lambda,l} - \\ & - \frac{1}{2} \sum_{\lambda=1} m_{1,l} + \sum_{\lambda=\pm 1} k n_{\lambda,k} \end{aligned}$$

Proof. The proof of the first statement is a trivial consequence of Section 5.5 in Bondal's paper. The formula (34) is (5.10) in [1] (with two small corrections: a factor 2 in the first term of the second row and the last term in the last row were missing). \square

In order to use this result to compute the dimension of our symplectic leaves we need to describe the Jordan normal form J_0 of $S^{-T}S$ for a Stokes matrix S with entries $s_{ij} = G_{ij}$ where G_{ij} are given either by (27) or by (29,30). This is achieved in the next two theorems which will be proved in subsections 4.2 and 4.3 respectively.

Theorem 4.3. *Let S be an upper triangular matrix with 1 on the diagonal and off diagonal entries*

$$S_{ij} = -\text{Tr}(\gamma_i \gamma_j), \quad i < j,$$

with $\gamma_1, \dots, \gamma_n$ given in terms of shear coordinates by formula (30). Then for n even, the matrix $S^{-T}S$, has the following Jordan form:

$$(35) \quad J_0 = \begin{pmatrix} -e^P & 0 & \mathbb{O} & \mathbb{O} \\ 0 & -e^{-P} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & -1 & 1 & \mathbb{O} \\ \mathbb{O} & 0 & -1 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & -\mathbb{I}_{n-4} \end{pmatrix},$$

while for n odd,

$$(36) \quad J_0 = \begin{pmatrix} e^P & 0 & 0 & \mathbb{O} \\ 0 & e^{-P} & 0 & \mathbb{O} \\ 0 & 0 & 1 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & -\mathbb{I}_{n-3} \end{pmatrix},$$

where $P = \sum_{i=1}^n Z_i + \sum_{j=1}^{n-3} Y_j$ is the central element corresponding to the face of the fat-graph.

Theorem 4.4. *Let S be an upper triangular matrix with 1 on the diagonal and off diagonal entries*

$$S_{ij} = -\text{Tr}(\gamma_i \gamma_j), \quad i < j,$$

with $\gamma_1, \dots, \gamma_n$ given in terms of shear coordinates by formula (39). Then the matrix $S^{-T}S$, has the following Jordan form for n even:

$$(37) \quad J_0 = \begin{pmatrix} -e^{P_1} & 0 & 0 & 0 & & \\ 0 & -e^{-P_1} & 0 & 0 & & \\ 0 & 0 & -e^{P_2} & 0 & & \\ 0 & 0 & 0 & -e^{-P_2} & & \\ & & \mathbb{O} & & & -\mathbb{I}_{n-4} \end{pmatrix},$$

and for n odd:

$$(38) \quad J_0 = \begin{pmatrix} e^P & 0 & 0 & 0 & 0 & \\ 0 & e^{-P} & 0 & 0 & 0 & \\ 0 & 0 & -1 & 1 & 0 & \mathbb{O} \\ 0 & 0 & 0 & -1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ & & \mathbb{O} & & & -\mathbb{I}_{n-5} \end{pmatrix},$$

where \mathbb{I}_{n-4} and \mathbb{I}_{n-5} are respectively the $(n-4) \times (n-4)$ and $(n-5) \times (n-5)$ identity matrices and $P_1 = \sum_{i=1}^n Z_i$, $P_2 = \sum_{j=1}^{2n-6} Y_j$ are the perimeters of the 2 holes in the case of n even, and $P = \sum_{i=1}^n Z_i + \sum_{j=1}^{2n-6} Y_j$ is the perimeter of the one hole for n odd.

Remark 4.5. Very similar Jordan normal forms appear for the matrix $S^{-T}S$ where S is the Stokes matrix associated to the Frobenius manifold structure on Hurwitz space (see Theorem 4 in [31]). However, in that case the central elements P or P_1, P_2 are rational multiples of $2\pi i$ rather than real numbers.

A first step in the direction of proving Theorems 4.3 and 4.4 is carried out in the next Lemma:

Lemma 4.6. The matrix of the symmetric form $G_{ij} = (S^T + S)_{ij}$ has at most rank four in the case of a Riemann surface $\Sigma_{g,s,0}$ of genus g with $s = 1, 2$ and at most rank three in the \mathcal{A}_n case.

Proof. We prove this lemma in the next subsection where an interesting interpretation in terms of n particle model in Minkowski space is studied. \square

4.1. Minkowski space model. In both CFP and \mathcal{A}_n cases, each element G_{ij} can be presented as $G_{ij} = -\text{Tr } \gamma_i \gamma_j$, where γ_k , $k = 1, \dots, n$, are given by (30) for the \mathcal{A}_n case and, thanks to Remark 3.1, by the following matrices in CFP case:

$$(39) \quad \begin{aligned} \gamma_1 &= F, \\ \gamma_2 &= -X_{\frac{Z_1}{2}} L X_{Z_2} R X_{\frac{Z_1}{2}} \\ \gamma_3 &= -X_{\frac{Z_1}{2}} R X_{Y_{n-2}} L X_{Z_3} R X_{Y_1} L X_{\frac{Z_1}{2}} \\ &\dots \\ \gamma_i &= -X_{\frac{Z_1}{2}} R X_{Y_{n-2}} R X_{Y_{n-1}} \dots R X_{Y_{n+i-5}} L X_{Z_i} R X_{Y_{i-2}} L \dots X_{Y_1} L X_{\frac{Z_1}{2}}, \\ &\dots \\ \gamma_{n-1} &= -X_{\frac{Z_1}{2}} R X_{Y_{n-2}} R X_{Y_2} \dots R X_{Y_{2n-6}} L X_{Z_{n-1}} R X_{Y_{n-3}} L \dots X_{Y_1} L X_{\frac{Z_1}{2}}, \\ \gamma_n &= -X_{\frac{Z_1}{2}} R X_{Y_{n-2}} R X_{Y_{n-1}} \dots R X_{Y_{2n-6}} R X_{Z_n} R X_{Y_{n-3}} L \dots X_{Y_1} L X_{\frac{Z_1}{2}}. \end{aligned}$$

Expand $\gamma_1, \dots, \gamma_n$ as

$$\gamma_i = \sum_{\alpha=1}^4 v_{\alpha}^{(i)} \sigma_{\alpha}, \quad i = 1, \dots, n,$$

where $\sigma_1, \dots, \sigma_4$ are the real Pauli matrices

$$\begin{aligned} \sigma_4 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \sigma_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

in the CFP case. In the latter case we have:

$$(40) \quad G_{ij} = v_1^{(i)} v_1^{(j)} - v_2^{(i)} v_2^{(j)} - v_3^{(i)} v_3^{(j)} - v_4^{(i)} v_4^{(j)} = \sum_{\alpha, \beta=1}^4 v_{\alpha}^{(i)} v_{\beta}^{(j)} \eta^{\alpha\beta},$$

where

$$(41) \quad \eta^{\alpha\beta} = \text{diag}(+, -, -, -)$$

is the metric tensor of the Minkowski 3 + 1-dimensional space-time.

In the \mathcal{A}_n case, because each γ_i is a conjugate of F , $\text{Tr} \gamma_i = 0$, no fourth component occurs. We then have

$$(42) \quad G_{ij} = v_1^{(i)} v_1^{(j)} - v_2^{(i)} v_2^{(j)} - v_3^{(i)} v_3^{(j)} = \sum_{\alpha, \beta=1}^3 v_{\alpha}^{(i)} v_{\beta}^{(j)} \eta^{\alpha\beta},$$

and $\eta^{\alpha\beta} = \text{diag}(+, -, -)$ is here the metric tensor of the Minkowski 2+1-dimensional space-time.

This concludes the proof of Lemma 4.6. \square

Remark 4.7. *It is interesting to notice that in the both cases, we can therefore associate $v_{\alpha}^{(i)}$ with the components of 4- or 3-dimensional vector $\mathbf{v}^{(i)}$ in the corresponding Minkowski space. Due to the fact that $\text{Tr} \gamma_i^2 = 2$ we obtain the restriction*

$$(43) \quad \|\mathbf{v}^{(i)}\|^2 \equiv v^{\alpha(i)} v_{\alpha}^{(i)} = (\mathbf{v}^{(i)}, \mathbf{v}^{(i)}) = 2 \quad \forall i,$$

where we use the standard repeated indices summation. This implies that all the vectors $\mathbf{v}^{(i)}$, $i = 1, \dots, n$ lie in the upper sheet of the hyperboloid of two sheets (they are time-like vectors in the physical terminology). In this case G_{ij} is the scalar product of the corresponding vectors,

$$(44) \quad G_{ij} = (\mathbf{v}^{(i)}, \mathbf{v}^{(j)})$$

and since the difference of two different time-like vectors lying on the same sheet is a space-like vector with negative norm, $\|\mathbf{v}^{(i)} - \mathbf{v}^{(j)}\|^2 = 4 - 2G_{ij} < 0$, and all G_{ij} are greater than two, as expected.

4.2. Proof of Theorems 4.3 and 4.1 in the \mathcal{A}_n case. We have proved in Lemma 4.6 that

$$\text{rk}(\mathcal{S}^{-T} \mathcal{S} + \mathbb{I}) = 3,$$

so we only need to compute the remaining 3 eigenvalues in order to prove Theorem 4.3. The proof is based on the following lemma:

Lemma 4.8. *All the eigenvalues of the matrix $\mathcal{S}^{-T}\mathcal{S}$ are functions of the only modular invariant parameter $P = \sum_{\alpha=1}^n Z_{\alpha} + \sum_{\beta=1}^{n-3} Y_{\beta}$, which is the sum of all the Teichmüller space variables.*

Proof. This is a simple consequence of the fact that the eigenvalues of the matrix $\mathcal{S}^{-T}\mathcal{S}$ are part of the monodromy data of the system (2) and therefore they must be central elements in the Dubrovin–Ugaglia bracket and therefore of the Goldman bracket (24). As a consequence the determinant of any linear combination $\lambda^{-1}\mathcal{S}^T + \lambda\mathcal{S}$ is a modular-invariant function. On the other hand, this determinant is a Laurent polynomial of order not higher than n in each of $e^{Z_{\alpha}/2}$, which inevitably means that this determinant is a Laurent polynomial of order not higher than n of $e^{P/2}$ alone. \square

The idea of the proof of Theorem 4.3 is to use the modular invariance to choose in a special way the parameters Z_i , $i = 2, \dots, n-1$ and Y_j , $j = 2, \dots, n-1$, leaving Z_1 and Z_n arbitrary. In fact, since the eigenvalues are modular invariants, if we change some of Z_i and Y_j by preserving their total sum, the eigenvalues must remain the same.

Because the determinant of $\lambda^{-1}\mathcal{S}^T + \lambda\mathcal{S}$ is a rational function in $e^{Z_{\alpha}/2}$ and $e^{Y_{\beta}/2}$, it has a unique analytic continuation in the domain of *complex values* of $Z_1, \dots, Z_n, Y_1, \dots, Y_{n-3}$. The value of the determinant must then be conserved provided the exponential e^P , $P = \sum_{\alpha=1}^n Z_{\alpha} + \sum_{\beta=1}^{n-3} Y_{\beta}$, remains invariant. We now present a convenient choice of these, complex, parameters. We take the representation graph (the spine) of the form depicted in Fig. 4, in which we specially indicated geodesic functions that will play an important role in the proof.

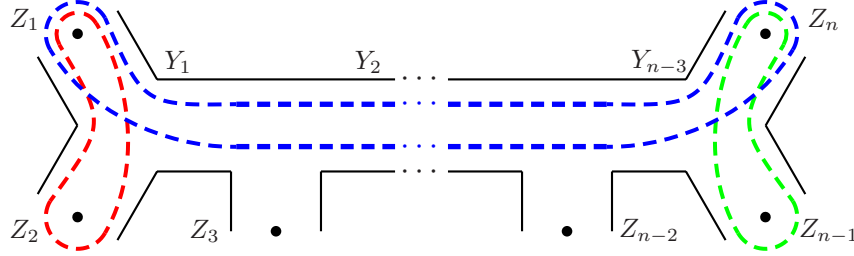


FIGURE 4. The fat graph for \mathcal{A}_n . The blue geodesic is $G_{1,n}$, the green one is $G_{n-1,n}$ and the red one is $G_{1,2}$.

We choose *all* the Y_j to be $-i\pi$, then $X_{Y_j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ for any $j = 1, \dots, n-3$. This special matrix is characterised by that $LX_Y L = RX_Y R = X_Y$. We also use extensively that $R = -L^2$ and $L = R^2$. We next take $Z_2 = -Z_3 = Z_4 = \dots = (-1)^{n-1}Z_{n-1}$ and leave Z_1 and Z_n arbitrary. Under this choice of the parameters, the entries $\tilde{G}_{ij} := S_{ij} + S_{ji}$ simplify considerably. Namely, we obtain

$$\begin{aligned}
 \tilde{G}_{1,2} &= -\tilde{G}_{13} = \tilde{G}_{14} = \dots = (-1)^{n-1}\tilde{G}_{1,n-1} = \text{Tr } LX_{2Z_2}RX_{2Z_1} \\
 \tilde{G}_{i,j} &= (-1)^{i-j}2, \quad 1 < i < j < n, \\
 \tilde{G}_{n-1,n} &= -\tilde{G}_{n-2,n} = \tilde{G}_{n-3,n} = \dots = (-1)^{n-1}\tilde{G}_{2,n} = \text{Tr } LX_{2Z_n}RX_{2Z_{n-1}} \\
 (45) \quad \tilde{G}_{1,n} &= \begin{cases} e^{Z_n+Z_1} + e^{-Z_n-Z_1}, & \text{even } n \\ \text{Tr } LX_{2Z_1}RX_{2Z_n} & \text{odd } n \end{cases}
 \end{aligned}$$

All the entries of the matrix \mathcal{S} are either ± 2 , or $\pm \tilde{G}_{1,2}$, or $\pm \tilde{G}_{n-1,n}$ or $\tilde{G}_{1,n}$. We are now going to show that we can re-arrange the rows and columns of the matrix $\lambda\mathcal{S} + \lambda^{-1}\mathcal{S}^T$ in order to obtain the form:

$$(46) \quad \det(\lambda\mathcal{S} + \lambda^{-1}\mathcal{S}^T) = \begin{vmatrix} \lambda + \lambda^{-1} & \lambda c & \lambda a & \lambda a & \cdots & \lambda a \\ \lambda^{-1}c & \lambda + \lambda^{-1} & \lambda b & \lambda b & \cdots & \lambda b \\ \lambda^{-1}a & \lambda^{-1}b & \lambda + \lambda^{-1} & 2\lambda & \cdots & 2\lambda \\ \lambda^{-1}a & \lambda^{-1}b & 2\lambda^{-1} & \lambda + \lambda^{-1} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 2\lambda \\ \lambda^{-1}a & \lambda^{-1}b & 2\lambda^{-1} & \cdots & 2\lambda^{-1} & \lambda + \lambda^{-1} \end{vmatrix}.$$

For such matrix form (46) we can easily compute the determinant:

$$(47) \quad \begin{aligned} \det(\lambda\mathcal{S} + \lambda^{-1}\mathcal{S}^T) &= [(\lambda + \lambda^{-1})^2 - c^2](\lambda - \lambda^{-1})^2 I_{n-4} \\ &+ (\lambda + \lambda^{-1})[(\lambda + \lambda^{-1})^2 + abc - a^2 - b^2 - c^2] I_{n-3}, \end{aligned}$$

where I_k is $(\lambda - \lambda^{-1})^k$ times the determinant of the skew-symmetric matrix with all the entries above the diagonal equal to the unity; this determinant is zero for odd k and 1 for even k . So, we obtain

$$(48) \quad \det(\lambda\mathcal{S} + \frac{\mathcal{S}^T}{\lambda}) = \begin{cases} [(\lambda + \lambda^{-1})^2 - c^2](\lambda - \lambda^{-1})^{n-2}, & \text{even } n, \\ (\lambda + \lambda^{-1})[(\lambda + \lambda^{-1})^2 + abc - a^2 - b^2 - c^2](\lambda - \lambda^{-1})^{n-3}, & \text{odd } n. \end{cases}$$

Let us prove formula (46) and deduce the values of the eigenvalues of J_0 in the even and in the odd dimensional cases separately.

For even n , we have that $\det(\lambda\mathcal{S} + \lambda^{-1}\mathcal{S}^T)$ is given by

$$\begin{vmatrix} \lambda + \lambda^{-1} & \lambda \tilde{G}_{1,2} & -\lambda \tilde{G}_{1,2} & \lambda \tilde{G}_{1,2} & \cdots & -\lambda \tilde{G}_{1,2} & \lambda \tilde{G}_{1,n} \\ \lambda^{-1} \tilde{G}_{1,2} & \lambda + \lambda^{-1} & -2\lambda & 2\lambda & \cdots & -2\lambda & -\lambda \tilde{G}_{n-1,n} \\ -\lambda^{-1} \tilde{G}_{1,2} & -2\lambda^{-1} & \lambda + \lambda^{-1} & -2\lambda & \cdots & 2\lambda & \lambda \tilde{G}_{n-1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda^{-1} \tilde{G}_{1,2} & 2\lambda^{-1} & \cdots & -2\lambda^{-1} & \lambda + \lambda^{-1} & -2\lambda & -\lambda \tilde{G}_{n-1,n} \\ -\lambda^{-1} \tilde{G}_{1,2} & -2\lambda^{-1} & \cdots & 2\lambda^{-1} & -2\lambda^{-1} & \lambda + \lambda^{-1} & \lambda \tilde{G}_{n-1,n} \\ \lambda^{-1} \tilde{G}_{1,n} & -\lambda^{-1} \tilde{G}_{n-1,n} & \cdots & \lambda^{-1} \tilde{G}_{n-1,n} & -\lambda^{-1} \tilde{G}_{n-1,n} & \lambda^{-1} \tilde{G}_{n-1,n} & \lambda + \lambda^{-1} \end{vmatrix}$$

and multiplying the odd columns and rows by -1 and cyclically permuting rows and columns $\{1, 2, \dots, n-1, n\} \rightarrow \{n, 1, 2, \dots, n-1\}$, we obtain the matrix of the form (46) with $a = -G_{n-1,n}$, $b = G_{1,2}$, and $c = G_{1,n}$. Neither a nor b however contribute to the determinant (48) for even n , whereas, from (45), $G_{1,n} = e^P + e^{-P}$ (because the contribution from other Z_i vanish for even n , $Z_2 + \cdots + Z_{n-1} = 0$). For even n we therefore have

$$(49) \quad \det(\lambda\mathcal{S} + \lambda^{-1}\mathcal{S}^T) = [(\lambda + \lambda^{-1})^2 - (e^P + e^{-P})^2](\lambda - \lambda^{-1})^{n-2},$$

and the roots of the characteristic equation $\det(\mathcal{S}^{-T}\mathcal{S} - \eta) = 0$ ($\eta = -\lambda^2$) are $\eta = \{-e^P, -e^{-P}, -1, \dots, -1\}$. Since the rank of $\mathcal{S}^{-T}\mathcal{S} + 1$ is less or equal three, the Jordan form (in the case of nonzero P) must have $n-2$ 1×1 blocks corresponding to the eigenvalues: $-e^P$, $-e^{-P}$, and $n-4$ eigenvalues -1 , and one 2×2 block $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. This concludes the proof of (35) for n even.

For odd n we have that $\det(\lambda S + \lambda^{-1} S^T)$ is given by

$$\begin{vmatrix} \lambda + \lambda^{-1} & \lambda \tilde{G}_{1,2} & -\lambda \tilde{G}_{1,2} & \lambda \tilde{G}_{1,2} & -\lambda G_{1,2} & \lambda \tilde{G}_{1,2} & \lambda \tilde{G}_{1,n} \\ \lambda^{-1} \tilde{G}_{1,2} & \lambda + \lambda^{-1} & -2\lambda & 2\lambda & -2\lambda & 2\lambda & \lambda \tilde{G}_{n-1,n} \\ -\lambda^{-1} \tilde{G}_{1,2} & -2\lambda^{-1} & \lambda + \lambda^{-1} & -2\lambda & 2\lambda & -2\lambda & -\lambda \tilde{G}_{n-1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\lambda^{-1} \tilde{G}_{1,2} & -2\lambda^{-1} & \cdots & 2\lambda^{-1} & -2\lambda^{-1} & -2\lambda & -\lambda \tilde{G}_{n-1,n} \\ \lambda^{-1} \tilde{G}_{1,2} & 2\lambda^{-1} & \cdots & -2\lambda^{-1} & 2\lambda^{-1} & \lambda + \lambda^{-1} & \lambda \tilde{G}_{n-1,n} \\ \lambda^{-1} \tilde{G}_{1,n} & \lambda^{-1} \tilde{G}_{n-1,n} & \cdots & -\lambda^{-1} \tilde{G}_{n-1,n} & \lambda^{-1} \tilde{G}_{n-1,n} & \lambda^{-1} \tilde{G}_{n-1,n} & \lambda + \lambda^{-1} \end{vmatrix}$$

and multiplying the odd columns and rows by -1 and cyclically permuting rows and columns $\{1, 2, \dots, n-1, n\} \rightarrow \{n, 1, 2, \dots, n-1\}$, we obtain the matrix of the form (46) with $a = \tilde{G}_{n-1,n}$, $b = G_{1,2}$, and $c = G_{1,n}$. Now, the elements $G_{1,2}$, $\tilde{G}_{n-1,n}$, and $G_{1,n}$ (see their explicit expressions in (45)) constitute the Markov triple, that is, $abc - a^2 - b^2 - c^2 = (e^P - e^{-P})^2$, where $P = Z_1 + Z_2 + Z_n$ is the perimeter of the hole (the remaining Z_i are mutually canceled). For odd n , we therefore have

$$(50) \quad \det(\lambda S + \lambda^{-1} S^T) = [(\lambda + \lambda^{-1})^2 + (e^P - e^{-P})^2](\lambda + \lambda^{-1})(\lambda - \lambda^{-1})^{n-3},$$

and the roots of the characteristic equation $\det(S^{-T} S - \eta) = 0$ ($\eta = -\lambda^2$) are now $\eta = \{e^P, e^{-P}, 1, -1, \dots, -1\}$. Since the rank of $S^{-T} S + 1$ is less or equal three, all these numbers are eigenvalues (for $P \neq 0$) and the Jordan form is diagonal. This concludes the proof of (36) for n odd. \square

4.2.1. *Symplectic leaves corresponding to \mathcal{A}_n .* We are now ready to prove that the dimension of the symplectic leaves $\mathcal{L}_{\mathcal{A}_n}$ corresponding to \mathcal{A}_n is

$$\dim_{\mathbb{C}}(L_{\mathcal{A}_n}) = 2(n-2)$$

which is the double the real dimension of the Teichmüller space.

Proof. Thanks to Theorem 4.3, for n even,

$$J_0 = n_{\lambda,1} (J_{\lambda,1} \oplus J_{\frac{1}{\lambda},1}) + m_{-1,2} J_{-1,2} + n_{-1,1} (J_{-1,1} \oplus J_{-1,1})$$

where

$$n_{\lambda,1} = 1, \quad m_{-1,2} = 1, \quad n_{-1,1} = \frac{n-4}{2}.$$

while for n odd,

$$J_0 = n_{\lambda,1} (J_{\lambda,1} \oplus J_{\frac{1}{\lambda},1}) + m_{1,1} J_{1,1} + n_{-1,1} (J_{-1,1} \oplus J_{-1,1})$$

where

$$n_{\lambda,1} = 1, \quad m_{1,1} = 1, \quad n_{-1,1} = \frac{n-3}{2}.$$

Using (34) we get precisely

$$d(S) = \frac{8 - 5n + n^2}{2} = \frac{n(n-1)}{2} - 2(n-2).$$

This concludes the proof of Theorem 4.1 in the \mathcal{A}_n case. \square

4.3. Proof of Theorems 4.4 and 4.1 in the CFP case. The idea of the proof is the same as for Theorem 4.3. We already proved in Lemma 4.6 that $\text{rk}(\mathcal{S}^{-T}\mathcal{S} + \mathbb{I}) = 4$, so we only need to compute the remaining 4 eigenvalues. Lemma 4.8 is still valid and we will now show how to pick the parameters Z_i , and Y_j , in a way to simplify computations.

Odd n . In this case, we have just one hole and we can set $e^{Y/2} = i$ and $Z_1 = -Z_2 = Z_3 = \dots = -Z_{n-1} = Z_n$ as before. Then, $G_{ij} = (-1)^{i-j}2$ for $i, j \neq 1, n$, $G_{1,2} = (-1)^j G_{1,j} = \text{Tr } LX_{-Z_1}RX_{Z_1}$ for $2 \leq j \leq n-1$, $G_{n-1,n} = (-1)^j G_{j,n} = \text{Tr } LX_{Z_1}RX_{-Z_1}$ for $2 \leq j \leq n-1$, and by multiplying the odd columns and rows by -1 and cyclically permuting rows and columns $\{1, 2, \dots, n-1, n\} \rightarrow \{n, 1, 2, \dots, n-1\}$, we obtain the matrix (46) with $a = G_{n-1,n}$, $b = G_{1,2}$, and $c = G_{1,n}$ having the same determinant (48).

The characteristic equation $\det(\mathcal{S}^{-T}\mathcal{S} - \eta\mathbb{I}) = 0$ where $\eta = -\lambda^2$, has $(n-3)$ -fold root $\eta = -1$ and three single roots $\varphi = 1$, $\varphi = e^P$, and $\varphi = e^{-P}$. Then, since the rank of the matrix $\mathcal{S}^{-T}\mathcal{S} + \mathbb{I}$ is four in the CFP case, we obtain formula (38).

Even n . In this case, we have two modular-invariant parameters P_1 and P_2 such that $\sum_{\alpha=1}^n Z_\alpha + \sum_{\beta=1}^{2n-6} Y_\beta = P_1 + P_2$. We cannot now set all the variables Y_j to be $i\pi$ in the graph in Fig. 2 because those are the variables that distinguish between the perimeters of these two holes. We can set however $Z_1 = -Z_2 = Z_3 = \dots = -Z_n$, take two of the variables Y , say, the variables Y_1 and Y_{n-2} of the two edges (above and below) that separate Z_2 and Z_3 to be arbitrary and set all the remaining Y_j to be $i\pi$ (and $X_Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$). We then have *six* basic matrix elements, $a = G_{1,n}$, $b = G_{2,n}$, $c = G_{3,n}$, $d = G_{1,2}$, $e = G_{1,3}$, and $f = G_{2,3}$, and the matrix $\lambda^{-1}\mathcal{S}^T + \lambda\mathcal{S}$ reduces by the same operations of row/column multiplication by -1 and cyclic permutations of row/columns to the form

$$\begin{aligned} \det & \begin{pmatrix} \lambda + \lambda^{-1} & a\lambda & b\lambda & c\lambda & c\lambda & \dots & c\lambda \\ a\lambda^{-1} & \lambda + \lambda^{-1} & d\lambda & e\lambda & e\lambda & \dots & e\lambda \\ b\lambda^{-1} & d\lambda^{-1} & \lambda + \lambda^{-1} & f\lambda & f\lambda & \dots & f\lambda \\ c\lambda^{-1} & e\lambda^{-1} & f\lambda^{-1} & \lambda + \lambda^{-1} & 2\lambda & \dots & 2\lambda \\ c\lambda^{-1} & e\lambda^{-1} & f\lambda^{-1} & 2\lambda^{-1} & \lambda + \lambda^{-1} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 2\lambda \\ c\lambda^{-1} & e\lambda^{-1} & f\lambda^{-1} & 2\lambda^{-1} & \dots & 2\lambda^{-1} & \lambda + \lambda^{-1} \end{pmatrix} \\ &= (\lambda - \lambda^{-1})^{n-4} \det \begin{pmatrix} \lambda + \lambda^{-1} & a\lambda & b\lambda & c\lambda \\ a\lambda^{-1} & \lambda + \lambda^{-1} & d\lambda & e\lambda \\ b\lambda^{-1} & d\lambda^{-1} & \lambda + \lambda^{-1} & f\lambda \\ c\lambda^{-1} & e\lambda^{-1} & f\lambda^{-1} & \lambda + \lambda^{-1} \end{pmatrix} \end{aligned}$$

We express the remaining determinant through two invariant determinants:

$$D_1 \equiv \det \begin{pmatrix} 2 & a & b & c \\ a & 2 & d & e \\ b & d & 2 & f \\ c & e & f & 2 \end{pmatrix} = (e^{P_1/2} + e^{-P_1/2} - e^{P_2/2} - e^{-P_2/2})^2, \text{ at } \lambda = \pm 1$$

and

$$D_2 \equiv \det \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = (e^{P_1/2} + e^{-P_1/2} + e^{P_2/2} + e^{-P_2/2})^2, \text{ at } \lambda = \pm i.$$

For the determinant in question, we have

$$\begin{aligned} & \det(\lambda^{-1}S^T + \lambda S) \\ &= (\lambda - \lambda^{-1})^{n-4} ((\lambda + \lambda^{-1})^2 (\lambda - \lambda^{-1})^2 + \frac{D_1}{4} (\lambda + \lambda^{-1})^2 - \frac{D_2}{4} (\lambda - \lambda^{-1})^2) \\ &= (\lambda - \lambda^{-1})^{n-4} \left((\lambda^2 - \lambda^{-2})^2 - 4 \cosh(P_1/2) \cosh(P_2/2) (\lambda^2 + \lambda^{-2}) \right. \\ (51) \quad & \left. + 4 \cosh^2(P_1/2) + 4 \cosh^2(P_2/2) \right) \end{aligned}$$

The roots of (51) for $\varphi = -\lambda^2$ are $n - 4$ -fold root $\varphi = -1$ and four simple roots $\varphi = -e^{(P_1+P_2)/2}$, $\varphi = -e^{-(P_1+P_2)/2}$, $\varphi = -e^{(P_1-P_2)/2}$, and $\varphi = -e^{-(P_1-P_2)/2}$. When all these roots are distinct, the Jordan form is diagonal and all the roots correspond to eigenvectors. This completes the analysis of the Jordan forms for the *CFP* case.

4.3.1. Symplectic leaves corresponding to the *CFP* case. We can now prove that the dimension of the symplectic leaves \mathcal{L}_{CFP} corresponding to the *CFP* case is

$$\dim_{\mathbb{C}}(L_{\mathcal{A}_n}) = \begin{cases} 3n - 7 & \text{for } n \text{ odd} \\ 3n - 8 & \text{for } n \text{ even} \end{cases}$$

which is the double the real dimension of the Teichmüller space.

Proof. Thanks to Theorem 4.4, we have that in the case of n even the Jordan normal form decomposes as:

$$J_0 = n_{\lambda_1,1} \left(J_{\lambda_1,1} \oplus J_{\frac{1}{\lambda_1},1} \right) + n_{\lambda_2,1} \left(J_{\lambda_2,1} \oplus J_{\frac{1}{\lambda_2},1} \right) + n_{-1,1} (J_{-1,1} \oplus J_{-1,1})$$

where

$$n_{\lambda_1,1} = 1, \quad n_{\lambda_2,1} = 1, \quad n_{-1,1} = \frac{m-4}{2},$$

and for n odd

$$J_0 = n_{\lambda,1} \left(J_{\lambda,1} \oplus J_{\frac{1}{\lambda},1} \right) + m_{-1,2} J_{-1,2} + m_{1,1} J_{1,1} + n_{-1,1} (J_{-1,1} \oplus J_{-1,1})$$

where

$$n_{\lambda,1} = 1, \quad m_{-1,2} = 1, \quad m_{1,1} = 1, \quad n_{-1,1} = \frac{m-5}{2}.$$

By using (34) we conclude the proof of Theorem 4.1 in the *CFP* case. \square

4.4. Complexification. In this section we observe that the Stokes matrices belonging to the Teichmüller symplectic leaves $\mathcal{L}_{\mathcal{A}_n}$ and \mathcal{L}_{CFP} can be parameterized in terms of *complex coordinates* $Z_1, \dots, Z_n, Y_1, \dots, Y_k$ where $k = n - 3$ in the \mathcal{A}_n case and $k = 2n - 6$ in the *CFP* case by the same formulae

$$S_{ij} = -\text{Tr} \gamma_i \gamma_j, \quad i < j,$$

where γ_i, γ_j are now matrices in $SL_2(\mathbb{C})$ still given by formulae (30) and (39) with complex $Z_1, \dots, Z_n, Y_1, \dots, Y_k$.

When the coordinates Z_i become complex, we can still use the same parameterization of elements of the discretely acting group, which becomes now a finitely generated subgroup of $PSL(2, \mathbb{C})$, not $PSL(2, \mathbb{R})$, i.e., a *Kleinian* group. This Kleinian group $\Delta_{g'} \subset PSL(2, \mathbb{C})$ describes now a handlebody, that is, the quotient of the upper half-space $\mathbb{H}_3^+ := \mathbb{C} \times \mathbb{R}^+$ by the action of $\Delta_{g'}$. The handlebody is geometrically a filled Riemann surface whose boundary is a closed Riemann surface of genus $g' = 2g + s - 1$ obtained from the action of this group on the boundary of \mathbb{H}_3^+ , i.e., on the complex plane \mathbb{C} , and admits a Schottky uniformisation.

Note that in this approach we *do not* present the three-dimensional manifold as a direct product of a Riemann surface (with holes) and a time interval; instead we have an actual handlebody endowed with the set of closed geodesics inside it; each closed geodesic corresponds, as before, to a conjugacy class of the Kleinian group.

Note that, in this case, we loose the distinction between holes and handles of the original Riemann surface $\Sigma_{g,s}$: if we consider two Riemann surfaces Σ_{g_1, s_1} and Σ_{g_2, s_2} such that they are described by the same number of shear coordinates, or in other words such that $\dim(\mathcal{T}_{g_1, s_1} \times \mathbb{R}^{s_1}) = \dim(\mathcal{T}_{g_2, s_2} \times \mathbb{R}^{s_2})$, they can be considered as different parameterisations of the *same* handle-body, as we demonstrate on the example below.

Example 4.9. Complexification of the Teichmüller space $\mathcal{T}_{1,1}$ of a torus with one hole and of the Teichmüller space $\mathcal{T}_{0,3}$ of a sphere with three holes.

In Fig. 5 the original (two-dimensional) Riemann surface is obtained under the action of a Kleinian group in \mathbb{H}_3^+ restricted to the real vertical slice \mathbb{H}_2^+ . Of course, this is possible only when the real slice of the Kleinian group is simultaneously a Fuchsian group itself, i.e., a discrete subgroup of $PSL(2, \mathbb{R})$. However, we can continuously vary the parameters X_i in the complex domain to ensure a smooth transition between two patterns, as shown in Fig. 5.

Note that on the intermediate stages of the transition process in Fig. 5 we have no embedded two-dimensional (geodesically closed) Riemann surface inside the handlebody; it is reconstructed only when the group again becomes Fuchsian.

Although the two Riemann surfaces in Fig. 5, $\Sigma_{1,1}$ and $\Sigma_{0,3}$, have different topologies, their sets of geodesic lengths are the same, so we say they are *isospectral*.

We introduce the set of (decorated) Teichmüller space coordinates Z_i , $i = 1, 2, 3$, for $\mathcal{T}_{1,1}$ and X_i , $i = 1, 2, 3$, for $\mathcal{T}_{0,3}$; then, in order for the spectra of geodesic functions to coincide, it suffices to make the identification (up to the action of the mapping class group in each of the surfaces)

$$(52) \quad e^{P_i/2} + e^{-P_i/2} = e^{Z_i/2 + Z_{i+1}/2} + e^{-Z_i/2 - Z_{i+1}/2} + e^{-Z_i/2 + Z_{i+1}/2}, \quad i = 1, 2, 3,$$

where $P_i = X_i + X_{i+1}$ are the perimeters of three holes of $\Sigma_{0,3}$ and the (standard) geodesic functions $G_{i,i+1}$ for $\Sigma_{1,1}$ stand in the right-hand sides.

Note that equations (52) not always admit real solutions in terms of Z_i for a given real X_i : the obstruction is provided by the Markov element,

$$(53) \quad \mathcal{M} = G_{1,2}G_{1,3}G_{2,3} - G_{1,2}^2 - G_{1,3}^2 - G_{2,3}^2.$$

In the case of the torus $\Sigma_{1,1}$ with real Z_i , we have the inequality $\mathcal{M} \geq 0$, whereas in the case of the sphere $\Sigma_{0,3}$ with real X_i , we have the inequality $\mathcal{M} \geq -4$, so Eqs. 52 admit real solutions both in Z_i and in X_i iff $\mathcal{M} \geq 0$.

In Fig. 6 we depict the explicit relation between the geodesic functions and indicate the image of the boundary curve. In Fig. 7 the same correspondence is

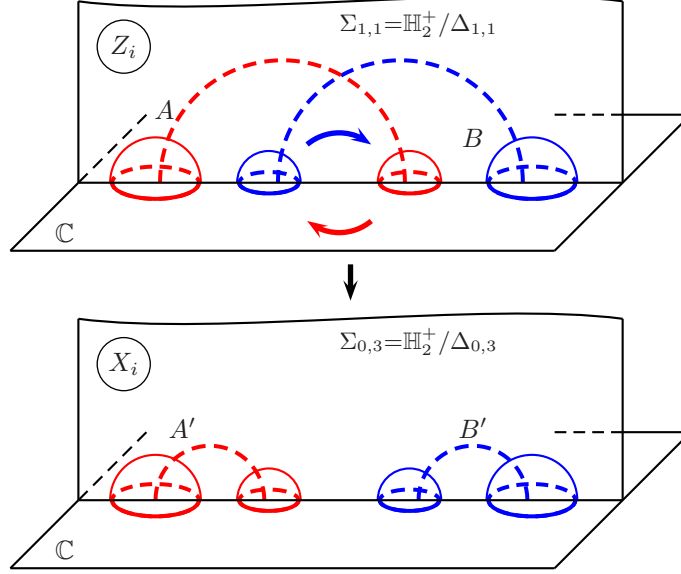


FIGURE 5. Deformation of the Fuchsian group transforming the torus with one hole into the sphere with three holes. We let A and B denote the geodesics corresponding to the respective A - and B -cycles on $\Sigma_{1,1}$; their images A' and B' are the geodesics corresponding to the perimeters of two of the holes in $\Sigma_{0,3}$; the perimeter of the third hole is $B'A'$.

presented for the spines $\Gamma_{1,1}$ and $\Gamma_{0,3}$. Note that neither the intersection indices between the curves nor the Poisson brackets are preserved under this identification.

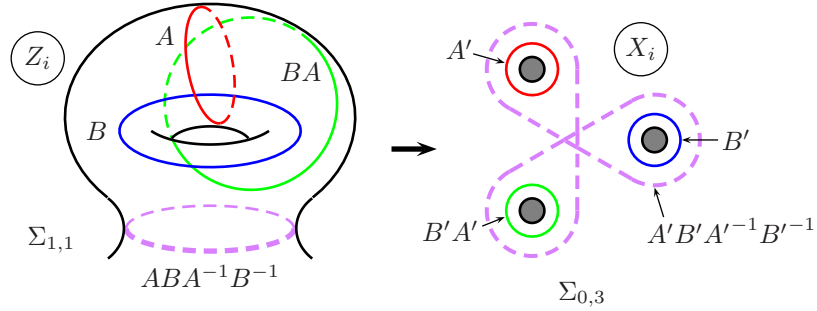


FIGURE 6. The transformation between geodesics on $\Sigma_{1,1}$ and $\Sigma_{0,3}$.

5. CONCLUSION

Theorems 4.3 and 4.4 characterise the Stokes matrices arising in the Teichmüller theory of a Riemann sphere with one hole and n orbifold points and of a Riemann surface of genus g with one or two holes respectively.

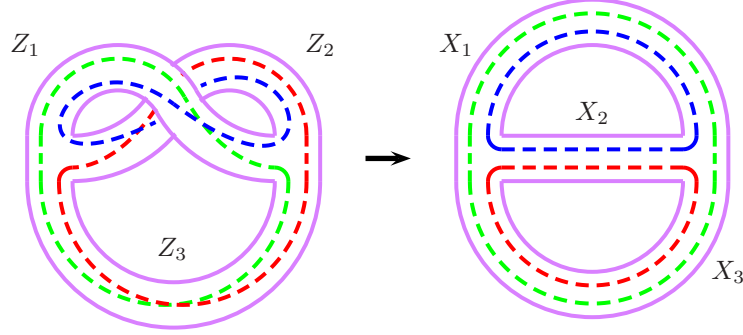


FIGURE 7. The transformation between geodesics in Fig. 6 depicted for the spines $\Gamma_{1,1}$ and $\Gamma_{0,3}$.

In section 4, we have seen that all Stokes matrices belonging to the degenerated symplectic leaves $\mathcal{L}_{\mathcal{A}_n}$ and \mathcal{L}_{CFP} can be parameterised in terms of complex coordinates $Z_1, \dots, Z_n, Y_1, \dots, Y_k$, where $k = n - 3$ in the \mathcal{A}_n case and $k = 2n - 6$ in the CFP case.

In order to characterise the Frobenius Manifold structure corresponding to these degenerated symplectic leaves one possible strategy is to determine the solution $V(u_1, \dots, u_n)$ of the isomonodromic deformation equation (13) and then to use Dubrovin's isomonodromicity theorem part III in [8] to reconstruct the metric, the flat coordinates, the pre-potential and the structure constants of the Frobenius manifold. Unfortunately at the moment this strategy fails at the very first step, i.e. we are unable to determine $V(u_1, \dots, u_n)$, even in the simplest case, i.e. for $n = 3$. We are going to explain what happens in this case in the next subsection and then in subsection 5.2 we will say a few words about the case of $n > 3$.

5.1. Case $n = 3$. For $n = 3$ we deal only with $\mathcal{L}_{\mathcal{A}_3}$ (the CFP case for $n = 3$ is completely equivalent to this one up to doubling of the shear coordinates). The geodesics G_{ij} are given by the following formula in which we use cyclic notation:

$$(54) \quad G_{i,i+1} = e^{Z_i + Z_{i+1}} + e^{Z_i - Z_{i+1}} + e^{-Z_i - Z_{i+1}},$$

which for $Z_1, Z_2, Z_3 \in \mathbb{R}$ are strictly bigger than 2. In this case all Stokes matrices in the generic symplectic leaves can be parameterised in terms of the complexified shear coordinates Z_1, Z_2, Z_3 , simply imposing

$$S = \begin{pmatrix} 1 & G_{1,2} & G_{3,1} \\ 0 & 1 & G_{2,3} \\ 0 & 0 & 1 \end{pmatrix},$$

where now $G_{i,j}$ are given by (54) with complex Z_1, Z_2, Z_3 .

For generic values of the central element $p = Z_1 + Z_2 + Z_3$, the Jordan normal form J_0 of the monodromy around 0 of system (2) is actually diagonal,

$$J_0 = \begin{pmatrix} e^p & 0 & 0 \\ 0 & e^{-p} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that the matrix V is non resonant and the Stokes matrix S determines uniquely the local solutions $V(u_1, u_2, u_3)$ of the isomonodromic deformation equations (13).

In this case the isomonodromic deformation equations reduce to a special case of the sixth Painlevé equation [7]

$$(55) \quad \begin{aligned} \ddot{y} &= \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \dot{y}^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \dot{y} + \\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left[\frac{(2\mu-1)^2}{2} + \frac{1}{2} \frac{t(t-1)}{(y-t)^2} \right], \end{aligned}$$

where $\mu = \frac{Z_1+Z_2+Z_3}{4i\pi}$, $t = \frac{u_2-u_1}{u_3-u_1}$ and

$$y = \frac{t(V_{12}V_{23} + \mu V_{13})^2}{(t-1)(\mu + V_{12}^2) + t(V_{12}V_{23} + \mu V_{13})^2},$$

so that the entries in the Stokes matrix uniquely determine the local solutions of this special case (55) of the sixth Painlevé equation.

The generic solutions of the sixth Painlevé equation are *irreducible* transcendental functions, i.e. they cannot be expressed via elementary or classical transcendental functions by simple operations. Of course some special solutions may be reducible: indeed all algebraic solutions of (55) were classified in [10] and [26], and the so called *classical solutions*, solutions that can be expressed in terms of hypergeometric functions were classified in [34]. However, in the geometric case, i.e. for $Z_1, Z_2, Z_3 \in \mathbb{R}$, the solutions are certainly irreducible: in [10] and [26] it was proved that in order to have algebraic solutions, a necessary condition is that $|S_{i,j}| < 2$, which is clearly violated in the geometric case. Moreover, using the results of [25], it is rather straightforward to prove that these solutions are never of hypergeometric type.

Another nasty surprise is given by looking at the asymptotic behaviour of the geometric solutions near the critical points. Indeed most PVI solutions have asymptotic behaviour of algebraic type, namely given σ_i , $i = 1, 2, 3$ complex numbers such that

$$2 \sin \frac{\pi \sigma_i}{2} = S_{jk}, \quad i \neq j, k, \quad \text{and} \quad \Re(\sigma_i) \in]0, 1[,$$

the corresponding PVI solution has the following asymptotic behaviours of algebraic type [23]:

$$y(t) \sim \begin{cases} a_0 t^{1-\sigma_3} (1 + \mathcal{O}(t)) & \text{for } t \rightarrow 0, \\ 1 - a_1 (1-t)^{1-\sigma_2} (1 + \mathcal{O}(1-t)) & \text{for } t \rightarrow 1, \\ a_\infty t^{\sigma_1} (1 + \mathcal{O}(\frac{1}{t})) & \text{for } t \rightarrow \infty. \end{cases}$$

However, for $S_{i,j} = G_{i,j} > 2$, we have $\sigma_i = 1 + i\nu_i$, $\nu_i \in \mathbb{R}$ for all $i = 1, 2, 3$. In this case the asymptotics are no longer of algebraic type, but become very complicated [18]. For example near 0 we have:

$$y(t) \sim \frac{1}{\sin^2 \left(\frac{\nu}{2} \log(x) + \phi + \frac{\nu}{2} F_1(x)/F(x) \right)},$$

where ϕ is a phase parameter and $F(x)$, $F_1(x)$ are the two Jacobi elliptic integrals. This makes all asymptotic computations of V and of the metric, the flat coordinates, the pre-potential and the structure constants of the Frobenius manifold extremely involved if not impossible.

5.2. Higher n . First observe that the discrepancy d between the dimension of the generic symplectic leaves and the dimension of the leaves $\mathcal{L}_{\mathcal{A}_n}$ and \mathcal{L}_{CFP} is given by:

$$d_{\mathcal{A}_n} := \dim(\mathcal{L}_{generic}) - \dim(\mathcal{L}_{\mathcal{A}_n}) = \begin{cases} \frac{1}{2}(n-3)^2 & \text{for } n \text{ odd} \\ \frac{1}{2}(n-2)(n-4) & \text{for } n \text{ even} \end{cases}$$

$$d_{CFP} := \dim(\mathcal{L}_{generic}) - \dim(\mathcal{L}_{CFP}) = \begin{cases} \frac{1}{2}(n-3)(n-5) & \text{for } n \text{ odd} \\ \frac{1}{2}(n-4)^2 & \text{for } n \text{ even} \end{cases}$$

we see that for $n = 3, 4$ the leaves $\mathcal{L}_{\mathcal{A}_n}$ are generic, while for $n = 4, 5$ the leaves \mathcal{L}_{CFP} are generic. As we have observed above, this fact is at the root of why we can't actually solve the isomonodromic deformation equations (13): for small n we deal with "generic solutions" which, as we have seen above, are irreducible transcendental functions.

For $n > 5$ the discrepancy d between the dimension of the generic symplectic leaves and the dimension of the leaves $\mathcal{L}_{\mathcal{A}_n}$ and \mathcal{L}_{CFP} is non zero. In terms of solutions $V(u_1, \dots, u_n)$ of the isomonodromic deformation equation (13), this means that the matrix function $V(u_1, \dots, u_n)$ satisfies extra d independent equations. These are algebraic equations that can be obtained by observing that as soon as n is large enough, J_0 has a block diagonal form in which one block is the minus identity. This means that V is resonant, and in principle we should have

$$J_0 = \exp(2\pi i \mu) \exp(2\pi i R),$$

where R is a nilpotent matrix satisfying (7) which can be recursively determined in terms of the entries of V . When a minus identity diagonal block appears, all off diagonal entries corresponding to that diagonal block must be zero, leading to extra equations for V . For example in the \mathcal{A}_n case, for n even we have $n - 2$ eigenvalues equal to -1 , so we should expect R to have $\frac{(n-2)(n-3)}{2}$ off diagonal entries. Since on our degenerated symplectic leaf only one of those is non zero, we expect $\frac{(n-4)(n-1)}{2}$ equations of which only $d = \frac{(n-4)^2}{2}$ are independent. Following the same train of thoughts as in [11], this implies that the solution $V(u_1, \dots, u_n)$ of the isomonodromic deformation equation (13) corresponding to the degenerated symplectic leaves can be in fact reduced to the Garnier system both in the \mathcal{A}_n and in the CFP case. Work on this reduction is still in progress.

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